BIRATIONAL GEOMETRY OF HYPERSURFACES IN PRODUCTS OF WEIGHTED PROJECTIVE SPACES

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We work / $k=\bar{k}$, char $k=0$, most of the times k uncountable.

All varieties are proj., normal, and $\mathbb{Q}$-factorial.

NOTATION/CONVENTIONS:

- $\underline{v}=(v_0,\dots,v_n)\in\mathbb{N}^{n+1}$. Grading on $k[x_0,\dots,x_n]$ as $\deg(x_i)=v_i$. The weighted proj. space of weights $\underline{v}$ is $\mathbb{P}^n(\underline{v}) := \mathrm{Proj}(k[x_0,\dots,x_n])$.
- All $\mathbb{P}^n(\underline{v})$ are assumed being well-formed, i.e. $\forall\, \{v_{i_1},\dots,v_{i_n}\}\subseteq\{v_0,\dots,v_n\}$ g.c.d.$\{v_{i_j}\}_j=1$.
- We set $e(\underline{v}) :=$ l.c.m.$\{v_i\}_i$.

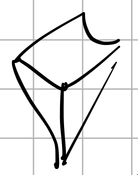
EXAMPLE: $\mathbb{P}^n(\underline{v}) = \mathbb{P}(\underbrace{1,\dots,1}_{n},d) \xhookrightarrow{\mathcal{O}(d)} \mathbb{P}^{h^0(\mathcal{O}(d))-1}$. Then it follows $\mathbb{P}^n(\underline{v})$ is cone over $V_d(\mathbb{P}^{n-1})$.

PROBLEM: Consider a hypersurface $Y \subseteq \prod_i \mathbb{P}^{n_i}(\underline{v}_i)$, what is the bir. geom. of Y?

RMK: In the smooth case the problem was approached by J. C. Ottem.

MORI DREAM SPACES:

We say that a variety X is a MDS if:

1) $h^1(X, \mathcal{O}_X) = 0$;
2) $\mathrm{Nef}(X)$ is rat. pol. () and generated by semiample divisor classes.
3) $\exists$ finitely many SQMs

$f_i : X \dashrightarrow X_i$ (i.e. X_i is normal & $\mathbb{Q}$-factorial, and f_i is an iso in codim 1) s.t.

$$\mathrm{Mov}(X) = \bigcup_i f_i^* \mathrm{Nef}(X_i)$$
$$\| $$
$$\mathrm{Cone}\{[D] \mid \mathrm{codim}(B(D)) \geq 2\}.$$

and any X_i satisfies (1), (2).

↑ GEOMETRIC VERSION

COX RINGS:

Suppose that X is a variety w/ $h^1(X, \mathcal{O}_X) = 0$ and $\mathrm{Cl}(X) = \mathrm{WDiv}(X)/\sim_{lin.}$ is free. Consider $\Gamma \subseteq \mathrm{WDiv}(X)$ a f.g. subgroup s.t. $\Gamma \xrightarrow{\sim} \mathrm{Cl}(X)$.

We define $\mathrm{Cox}(X) := \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D))$
$\ni s$

Fact (Hu-Keel): X w/ $h^1(X, \mathcal{O}_X) = 0$ is a MDS iff $\mathrm{Cox}(X)$ is of finite type / $\mathbb{C}$.

(NON)-EXAMPLES:

(1) Fano varieties (X smooth, $-K_X$ ample) are MDSs (by BCHM).

(2) A very general surface $S \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,3)|$ is a MDS.

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,3) = p_1^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(3)$$

(3) $S_4 := \left\{ \sum_{i=0}^{3} x_i^4 = 0 \right\} \subset \mathbb{P}^3_{\mathbb{C}}$ is not a MDS, because $\mathrm{Nef}(S_4)$ is not f.g.

(4) $X_3 \subseteq \mathbb{P}^5$ smooth cubic 4-fold s.t.

$$H^4(X_3, \mathbb{Z}) \cap H^{2,2}(X_3, \mathbb{C}) = \mathbb{Z} h^2 + \mathbb{Z}[T],$$

$T \subseteq X_3$ cubic scroll, h hyperplane class.

Then $F_1(X_3)$ is not a MDS; need ∞ many SQMs to recover $\mathrm{Mov}(F_1(X_3))$

HYPERSURFACES IN $\mathbb{P}^n(\underline{v}) \times \mathbb{P}^m(\underline{w})$

We consider ample Cartier hypersurfaces Y in $\mathbb{P}^n(\underline{v}) \times \mathbb{P}^m(\underline{w}) =: X$.

PROPOSITION: X, Y as above. If $n, m \geq 2$ then Y general in its linear system is a MDS and

$$\mathrm{Cox}(Y) = \mathrm{Cox}(X)/(f),$$ where

$Y = \{f = 0\}$ in X, and $\mathrm{Cox}(X) = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$.

SUPPOSE now $X = \mathbb{P}^1 \times \mathbb{P}^m(\underline{w})$, $m \geq 3$.

THEOREM (O.):

(1) Y general in $|\mathcal{O}_X(d, e)|$ is a MDS when $1 \leq d \leq m$. In particular

(a) $1 < d < m$, Y admits one and only one SQM, and a fibration onto $\mathbb{P}^1$.

(b) $d = 1$, Y admits a divisorial contraction and a fibration onto $\mathbb{P}^1$.

(c) $d = n$, Y admits 2 fibrations: one onto $\mathbb{P}^1$ and one onto $\mathbb{P}^{d-1}$.

In this case $\mathrm{Cox}(Y) \cong \dfrac{\mathbb{C}[x_0, x_1, y_0, \dots, y_n, z_1, \dots, z_n]}{I}$

$$I = (f_0 + x_0 z_1, f_1 - x_0 z_1 + x_1 z_2, \dots, f_{d-1} - x_0 z_{d-1}, f_d - x_0 z_d)$$

where the f_i are s.t.

$$Y = \{x_0^d f_0 + x_0^{d-1} x_1 f_1 + \dots + x_1^d f_d = 0\}$$

(2) If $d > n$, $e \geq 2 \cdot e(\underline{W})$, then the very general member $Y \in |\mathcal{O}_X(d, e)|$ is not a MDS: there is a nef divisor of negative Iitaka dim.

(3) The special member Y of $|\mathcal{O}_X(d, a(\underline{w}))|$ is a MDS if $d > m$.

Idea of Pf of (1.(a)): Write Y as a deg. locus of morph of vector bundles

$$Y = \{\det(M) = 0\}, \quad M = \begin{pmatrix} x_1 & -x_0 & \cdots & 0 \\ 0 & x_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & x_1 & -x_0 \\ f_0 & f_1 & \cdots\ f_{d-1} & f_d \end{pmatrix}$$

$$M : \mathcal{O}_X^{d+1} \longrightarrow \mathcal{O}_X^{d} \oplus \mathcal{O}_X(0,c)$$

Observe $\mathrm{rk}(M)$ decreases exactly of 1 at pts of Y.

If $p \in Y$ $\exists!$ $(z_1, \dots, z_d, 1)^t \in \ker(M(p))$.

We can define

$$g : Y \dashrightarrow \mathbb{P}^{d-1} \times \mathbb{P}^m(\underline{w}) = X'$$

$$((x_0 : x_1), (y_0 : \dots : y_m)) \longmapsto ((z_1 : \dots : z_d), (y_0 : \dots : y_m))$$

$\quad\ \ /\!/ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ /\!/$

$\quad\ p \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ q$

g is not defined at p s.t.

$\{ f_i(p) = 0 \}_i$.

We note that q is s.t. $\mathrm{rk}(M'(q)) = 2$,

$$M' = \begin{pmatrix} 0 & -\partial_1 & \cdots & \partial_{d-1} & -\partial_d \\ \partial_1 & \partial_2 & \cdots & -\partial_d & 0 \\ f_0 & f_1 & \cdots & f_{d-1} & f_d \end{pmatrix}$$

$$M' : \mathcal{O}_{X'}^{d+1} \longrightarrow \mathcal{O}_{X'}(1,0)^d \oplus \mathcal{O}_{X'}(0,e).$$

So if $Y' = \left\{ q \in \mathbb{P}^{d-1} \times \mathbb{P}^m (w) = X' \;\middle|\; \mathrm{rk}(M'(q)) = 2 \right\}$

We obtain $Y \dashrightarrow Y'$.

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Y' has a natural scheme structure.

We have to show that for Y gen. Y' is a normal $\mathbb{Q}$-fact proj. var.

AS FOR THE COX RING:

Since $\mathrm{rk}(M)$ decreases of exactly 1 at $p \in Y$, we obtain $Y \hookrightarrow \mathbb{P}(\mathcal{E})$,

$$\mathcal{E} = \mathcal{O}_X(1,0)^d \oplus \mathcal{O}_X(0,e).$$

If $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the structural morph, we can compute that

$$\mathrm{Cox}(\mathbb{P}(\mathcal{E})) \cong \mathbb{C}[x_0, x_1, y_0, \ldots, y_m, z_1, \ldots, z_d, t]$$

$x_i \in H^0(\pi^* \mathcal{O}_X(1,0))$, $y_i \in H^0(\pi^* \mathcal{O}_X(0, w_i))$, $z_i \in H^0(\pi^* \mathcal{O}_X(-1, e))$, $t \in H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}(0,-e))$

$Z(t)$ is the exc. locus of a div. cont. of $\mathbb{P}(\mathcal{E})$.

The generators of $I + t$ are the equations of Y in $\mathbb{P}(\mathcal{E})$.

<u>Idea of Pf of (2)</u>: Consider the morphism $\mathbb{P}^1 \times \mathbb{P}^m(\underline{w}) \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^m$ s.t.

$$(x_0 : x_1), (y_0 : \ldots : y_m) \mapsto (x_0 : x_1), (y_0^{e(\underline{w})/w_0} : \ldots : y_m^{e(\underline{w})/w_m}).$$

Note that if $L = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^m}(d, e/e(\underline{w}))|$,

$f^{*} L = V \subseteq |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^m(\underline{w})}(d, e)|$.

By Ottem's results, the very gen. member of L is not a MDS. Using this, along with f we find $Y_0 \in V$ s.t.

$[m e H_2 - d H_1$, has Iitaka dim < 0

not $\Rightarrow Y_0$ not MDS. We conclude with a defo argument that the very gen. member of $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^m(\underline{w})}(d, e)|$ is not a MDS.

Rem: There is an analogous res. of THM for surfaces in $\mathbb{P}^1 \times \mathbb{P}^2(\underline{w})$ s.t. $\underline{w} = (1,1,2)$ or $\underline{w} = (1,2,3)$ iff $\mathbb{P}^2(\underline{w})$ is Gorenstein.

WHAT ABOUT $Y \subseteq \prod_i \mathbb{P}^{m_i}(\underline{w}^i)$?

THM (d.) Suppose $X = (\mathbb{P}^1)^n \times \mathbb{P}^{m_1}(\underline{w}^1) \times \dots \times \mathbb{P}^{m_k}(\underline{w}^k)$.

$\to \operatorname{Spec}(\mathbb{C})$, X Gorenstein, with $\dim(X) \geq 4$ and $m_i \geq 2\ \forall i$.

(1) $n = 0, 1$, Y general $|-K_X|$ is a MDS.

(2) $n \geq 2 \Rightarrow Y$ general in $|-K_X|$ is NOT a MDS.

(3) X terminal $\Rightarrow Y$ general in $|-K_X|$ satisfies the birational version of the Kawamata - Morrison - Totaro cone conjecture.